The homotopy dimension of codiscrete subsets of the 2-sphere $\mathbf{S}^2$

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Abstract. Andreas Zastrow conjectured, and Cannon-Conner-Zastrow proved, (see [3,pp. 44-45]) that filling one hole in the Sierpinski curve with a disk results in a planar Peano continuum that is not homotopy equivalent to a 1-dimensional set. Zastrow's example is the motivation for this paper, where we characterize those planar Peano continua that are homotopy equivalent to 1-dimensional sets.

While many planar Peano continua are not homotopically 1-dimensional, we prove that each has fundamental group that embeds in the fundamental group of a 1-dimensional planar Peano continuum.

We leave open the following question: Is a planar Peano continuum homotopically 1-dimensional if its fundamental group is isomorphic with the fundamental group of a 1-dimensional planar Peano continuum?

1. Introduction. We say that a subset X of the 2-sphere $\mathbf{S}^2$ is *codiscrete* if and only if its complement $D(X)$, as subspace of $\mathbf{S}^2$, is discrete. The set $B(X)$ of limit points of $D(X)$ in $\mathbf{S}^2$, which is necessarily a closed subset of X having dimension ≤ 1, is called the *bad set* of X. Our main theorem characterizes the homotopy dimension of X in terms of the interplay between $D(X)$ and $B(X)$:

Characterization Theorem 1.1. Suppose that X is a codiscrete subset of the 2-sphere $\mathbf{S}^2$. Then X is homotopically at most 1-dimensional if and only if the following two conditions are satisfied.

(1) Every component of $\mathbf{S}^2 \setminus B(X)$ contains a point of $D(X)$.

(2) If D is any closed disk in the 2-sphere $\mathbf{S}^2$, then the components of $D \setminus B(X)$ that do not contain any point of $D(X)$ form a null sequence.

[Recall that a sequence $C_1, C_2, \ldots$ is a *null sequence* if the diameters of the sets C_n approach 0 as n approaches ∞.] Examples appear in Figures 1 and 2. Figure 1 gives two examples of possible bad sets that are locally connected. The one is a circle with countably many copies of the Hawaiian earring attached. The other is a Sierpinski curve. The associated codiscrete set will be homotopically 1-dimensional if and only if condition (1) is satisfied. Figure 2 gives an example of a possible bad set that is not locally connected. In order that the associated codiscrete set be homotopically 1-dimensional, both conditions (1) and (2) must be satisfied. Thus there must be points of the discrete set near each point of the bad set on both local sides of the bad set near the vertical limiting arc.

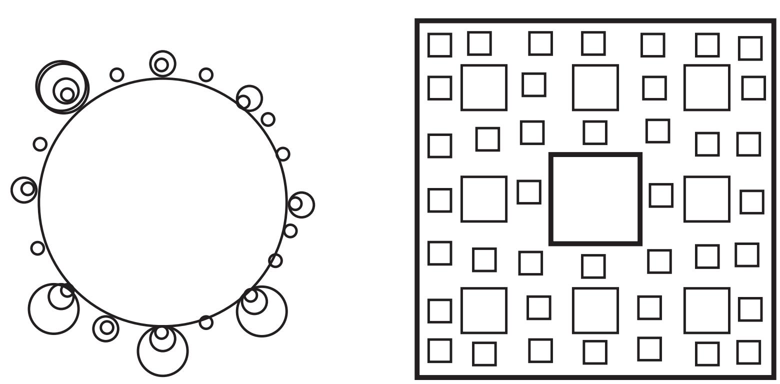

Figure 1. Possible bad sets that are locally connected.

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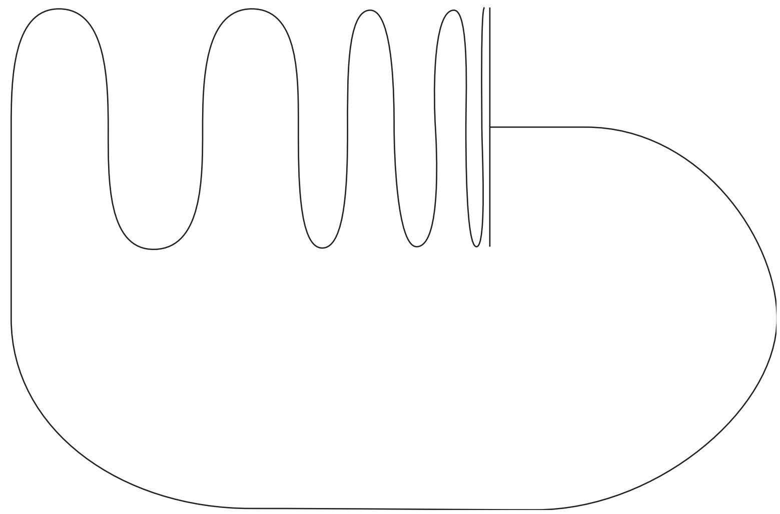

Figure 2. A possible bad set that is not locally connected.

A *continuum* is a compact, connected metric space. A *Peano continuum* is a locally connected continuum; equivalently, a Peano continuum is the metric continuous image of the interval $[0, 1]$. Characterization Theorem 1.1 applies to all Peano continua in the 2-sphere $\mathbf{S}^2$ because of the following well-known theorem:

Theorem 1.2. Every Peano continuum M in the 2-sphere $\mathbf{S}^2$ is homotopy equivalent to a codiscrete subset X of $\mathbf{S}^2$. Conversely, every codiscrete subset X of $\mathbf{S}^2$ is homotopy equivalent to a Peano continuum M in $\mathbf{S}^2$.

We shall indicate later [after Theorem 2.4.2] how this well-known theorem is proved. For the moment, we simply mention that, given M, one can obtain an appropriate codiscrete subset X by choosing for $D(X)$ exactly one point from each component of $\mathbf{S}^2 \setminus M$. One can define the bad set $B(M)$ of M as the bad set $B(X)$ of X. It is natural to ask how restricted bad sets are. The following theorem, which characterizes the possible bad sets of codiscrete sets X, is actually an easy exercise which we leave to the reader:

Theorem 1.3. A subset B of the 2-sphere $\mathbf{S}^2$ is the bad set $B(X)$ of some codiscrete subset $X \subset \mathbf{S}^2$ if and only if B is closed and has dimension less than 2.

It is an easy matter to use Characterization Theorem 1.1 and the construction inherent in Theorem 1.3 to construct all manner of interesting planar Peano continua that are, or are not, homotopy equivalent to a 1-dimensional set. All the examples that have appeared in the literature (see [3] and [8]) are likewise easily checked by means of Characterization Theorem 1.1.

In light of the fact that so many planar Peano continua are not homotopically 1-dimensional, it is a little surprising to find that their fundamental groups are essentially 1-dimensional in the following sense:

Theorem 1.4. If M is a planar Peano continuum, then the fundamental group of M embeds in the fundamental group of a 1-dimensional planar Peano continuum.

Corollary 1.4.1. If M is a planar Peano continuum, then the fundamental group of M embeds in an inverse limit of finitely generated free groups.

Question 1.4.2. If M is a planar Peano continuum whose fundamental group is isomorphic with the fundamental group of some 1-dimensional planar Peano continuum, is it true that M is homotopically 1-dimensional?

The remaining sections of this paper will be devoted to proofs of these theorems.

2. Fundamental ideas and tools. We collect here the basic ideas and tools that will be used often in the proofs. Many of these will be familiar to some of our readers. The topics will be outlined in bold type so that the reader can quickly find those topics with which they are not familiar. For many, the best way to read the paper will be to turn immediately to the later sections and return to this section only when they encounter a tool or idea with which they are not familiar.

Our first fundamental idea is that **homotopies of X within itself must fix the bad set $B(X)$ pointwise**. This general principle can be applied to all connected planar sets X and not just to codiscrete sets. If X is any connected planar set, then we may define the *bad set* $B(X)$ of X to be the set of points $x \in X$ having the property that, in each neighborhood of x there is a simple closed curve J in X such that the interior of J in the plane $\mathbf{R}^2$ is not entirely contained in the set X.

Theorem 2.1. Suppose that X is a connected planar set and that $x \in B(X)$. Then every homotopy of X within X fixes the point x.

Proof. Suppose that there is a homotopy $H : X \times [0,1] \to X$ such that $\forall y \in X$, $H(y,0) = y$ and such that $H(x,1) \neq x$. Let N_0 and N_1 be disjoint neighborhoods of x and $H(x,1)$, respectively. By continuity, there is a neighborhood M of x in N_0 such that $H(M,1) \subset N_1$. There is a round circle J around x that is not contained in X but intersects X only in N_0. There is a simple closed curve K in $\mathrm{int}(J) \cap M \subset X$ whose interior is not contained entirely in X. The annulus $H(K \times [0,1])$ has its boundary components separated by some component of $H^{-1}(J \cap X)$. That component maps into a single component of $J \cap X$, where it can be filled in via the Tietze Extension Theorem. This allows one to shrink K in X, an impossibility.

Our second fundamental idea is that of the **convergence of a sequence of sets**. Suppose that $A_1, A_2, \ldots$ is a sequence of subsets of a space S. We say that a point $x \in S$ is an element of $\liminf_i(A_i)$ if every neighborhood of x intersects all but finitely many of the sets A_i. We say that x is an element of $\limsup_i(A_i)$ if every neighborhood of x intersects infinitely many of the sets A_i. We say that the sequence A_i converges if the lim inf and lim sup coincide. The limit is defined to be this common lim inf and lim sup. Here are the fundamental facts about convergence, all of them well-known:

Theorem 2.2.1. If $A_1, A_2, \ldots$ is any sequence of sets in a separable metric space S, then there is a convergent subsequence.

Proof. Let $U_1, U_2, \ldots$ be a countable basis for the topology of S. Let S_0 be the given sequence $A_1, A_2, \ldots$ of subsets of the space S. Assume inductively that a subsequence S_i of S has been chosen. If there is a subsequence of S_i no element of which intersects U_{i+1}, let S_{i+1} be such a subsequence. Otherwise, let $S_{i+1} = S_i$. Let S_∞ be the diagonal sequence, which takes as first element the first element of S_1, as second element the second element of S_2, etc. We claim that the subsequence S_∞ of S_0 converges. Indeed, suppose that $x \in \limsup S_\infty$ That is, every neighborhood of x intersects infinitely many elements of S_∞. Suppose that there is a neighborhood U_j of x that misses infinitely many elements of S_∞. Then S_j, by definition, must miss U_j. But this implies that all elements of S_∞ with index as high as j miss U_j, a contradiction. Thus, every element of the lim sup lies in the lim inf. Since the opposite inclusion is obvious, these two limits are equal; and the sequence S_∞ converges.

Theorem 2.2.2. Properties of the limit of a convergent sequence. Suppose that the sequence $A_1, A_2, \ldots$ of nonempty subsets of a separable metric space S converges to a set A. Then,

(1) the set A is closed in S;

(2) if S is compact, then A is nonempty and compact;

(3) If S is compact and if each A_i is connected, then the limit A is nonempty, compact, and connected.

(4) If S is compact and if each A_i has diameter $\geq \epsilon$, then A has diameter $\geq \epsilon$.

Proof. Easy exercise.

We shall in more than one place make use of **R. L. Moore's Decomposition Theorem**. In 1919 [9], R. L. Moore characterized the Euclidean plane topologically. In 1925 [10], he noted that his axioms were also satisfied by a large class of quotient spaces of the plane, so that those identification spaces were also planes.

Since Moore's theorem is somewhat inaccessible to today's readers because of evolving terminology and background, we will give a fairly straightforward statement and outline the proof of this theorem.

Moore Decomposition Theorem 2.3.1. Suppose that $f : \mathbf{S}^2 \to X$ is a continuous map from the 2-sphere $\mathbf{S}^2$ onto a Hausdorff space X such that, for each $x \in X$, the set $\mathbf{S}^2 \setminus f^{-1}(x)$ is homeomorphic with the plane $\mathbf{R}^2$. Then X is a 2-sphere.

Remarks. (1) The requirement that $\mathbf{S}^2 \setminus f^{-1}(x)$ be homeomorphic with $\mathbf{R}^2$ is equivalent to the require-

ment that both $f^{-1}(x)$ and $\mathbf{S}^2 \setminus f^{-1}(x)$ be nonempty and connected.

(2) The theorem has the following generalization to higher dimensions: *Suppose that $f : \mathbf{S}^n \to X$ is a continuous map from the n-sphere $\mathbf{S}^n$ onto a Hausdorff space X such that, for each $x \in X$, the set $\mathbf{S}^n \setminus f^{-1}(x)$ is homeomorphic with the Euclidean space $\mathbf{R}^n$. Then X is an n-sphere provided that, in addition, $n \geq 5$, and X satisfies the condition that maps $g : \mathbf{B}^2 \to X$ from the 2-dimensional disk $\mathbf{B}^2$ into X can be approximated by embeddings.* This generalization was conjectured and proved in many special cases by Cannon (see [1] for a substantial discussion of these matters) and proved in general by R. D. Edwards (see Daverman's book [5].) The situation in dimensions 3 and 4 has not been completely resolved.

The proof we shall give relies on a more intuitive theorem, called the Zippin Characterization Theorem. (See, for example, [13, p. 88].)

Zippin Characterization Theorem 2.3.2. The space X is a 2-sphere if the following four conditions are satisfied:

(i) X is a nondegenerate Peano continuum.

(ii) No point $x \in X$ separates X (so that, in particular, X contains at least one simple closed curve).

(iii) Each simple closed curve $J \subset X$ separates X.

(iv) No arc $A \subset X$ separates X.

Proof of the Moore Decomposition Theorem on the basis of the Zippin Characterization Theorem. We verify the four conditions of the Zippin Theorem in turn. (Note that conditions (iii) and (iv) are true in the 2-sphere by standard homological arguments. We shall use those same arguments here.)

(i): Since X is Hausdorff, the map f is a closed surjection; hence it is easy to verify the conditions of the Urysohn metrization theorem so that X is metric. (See [11, Theorem 34.1].) Since $\mathbf{S}^2$ is a Peano continuum, that is, a metric continuous image of $[0, 1]$, so also is X. Since, $\forall x \in X$, both $f^{-1}(x)$ and $\mathbf{S}^2 \setminus f^{-1}(x)$ are nonempty, X has more than one point; that is, X is nondegenerate.

(ii): By hypothesis, $\mathbf{S}^2 \setminus f^{-1}(x)$ is connected. Hence $X \setminus \{x\} = f(\mathbf{S}^2 \setminus f^{-1}(x))$ is also connected.

(iii): Let $p_1, p_2 \in J$ cut J into two arcs A_1 and A_2. Then $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are compact, connected, and have nonconnected intersection $f^{-1}(p_1) \cup f^{-1}(p_2)$. The reduced Mayer-Vietoris homology sequence for the pair $U = \mathbf{S}^2 \setminus f^{-1}(A_1)$ and $V = \mathbf{S}^2 \setminus f^{-1}(A_2)$ contains the segment

$$H_1(\mathbf{S}^2 \setminus f^{-1}(A_1)) \oplus H_1(\mathbf{S}^2 \setminus f^{-1}(A_2)) \to H_1(\mathbf{S}^2 \setminus (f^{-1}(p_1) \cup f^{-1}(p_2)) \to \tilde{H}_0(\mathbf{S}^2 \setminus (f^{-1}(J)),$$

where $H_1(U) = H_1(V) = 0$ since $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are connected and $H_1(U \cup V) \neq 0$ since $f^{-1}(A_1) \cap f^{-1}(A_2)$ is not connected. Thus $\tilde{H}_0(\mathbf{S}^2 \setminus f^{-1}(J)) = \tilde{H}_0(U \cap V) \neq 0$, so that $f^{-1}(J)$ separates $\mathbf{S}^2$. Consequently, J separates X.

(iv): If $p \in A$ separates A into arcs A_1 and A_2, and if A separates x and y in X, then we claim that one of A_1 and A_2 also separates x and y in X; indeed, we see this by considering $f^{-1}(A) = f^{-1}(A_1) \cup f^{-1}(A_2)$, which must separate $f^{-1}(x)$ from $f^{-1}(y)$ in $\mathbf{S}^2$. The reduced Mayer-Vietoris homology sequence for the pair $U = \mathbf{S}^2 \setminus f^{-1}(A_1)$ and $V = \mathbf{S}^2 \setminus f^{-1}(A_2)$ contains the segment

$$0 \to \tilde{H}_0(\mathbf{S}^2 \setminus f^{-1}(A)) \to \tilde{H}_0(\mathbf{S}^2 \setminus A_1) \oplus \tilde{H}_0(\mathbf{S}^2 \setminus A_2).$$

The element $x - y$ represents a nonzero element of the center group, hence maps to a nonzero element of $\tilde{H}_0(\mathbf{S}^2 \setminus A_1) \oplus \tilde{H}_0(\mathbf{S}^2 \setminus A_2)$, as desired.

By induction, one obtains intervals $I_0 \supset I_1 \supset \cdots$ that separate x and y in X such that $\cap_{n=1}^{\infty} I_n$ is a single point q that does not separate x from y. But an arc α from x to y in the path connected open set $X \setminus \{q\}$ misses some I_n, a contradiction. We conclude that A cannot separate X.

The proof of the Moore Decomposition Theorem 2.3.1 is complete.

Our fourth topic is that of **locally connected continua in the plane**.

Theorem 2.4.1. Suppose that M is a continuum (= compact, connected subset) in the 2-sphere $\mathbf{S}^2$. Then M is a Peano continuum (= locally connected continuum) if and only if the following four equivalent conditions are satisfied.

(1) For each disk D in $\mathbf{S}^2$, the components of $D \setminus M$ form a null sequence.

(1′) For each disk D in $\mathbf{S}^2$, the components of $D \cap M$ form a null sequence.

(2) For each annulus A in $\mathbf{S}^2$, the components of $A \setminus M$ that intersect both boundary components of A are finite in number.

(2′) For each annulus A in $\mathbf{S}^2$, the components of $A \cap M$ that intersect both boundary components of A are finite in number.

Proof. Assume that M is locally connected but that (1) is not satisfied, so that, for some disk D in $\mathbf{S}^2$, the components of $D \setminus M$ do not form a null sequence. Then some sequence U_i of such components converges to a nondegenerate continuum U in $\mathbf{S}^2$ by Theorems 2.2.1 and 2.2.2. Let A be an annulus in $\mathbf{S}^2$ that separates two points of U. Then each U_i contains an arc A_i irreducibly joining the two ends of A. We may assume that they converge to a continuum A' joining the two ends of A. The continuum A' must be a subset of M, for otherwise it could not have points of infinitely many of the components U_i close to it. Since the arcs A_i converge to A', there must be two of them, which we may number as A_1 and A_2, that have no other A_i nor A' between them. There are then only two components of $A \setminus (A' \cup A_1 \cup A_2)$ that can contain any of the remaining A_i. This allows us to choose a subsequence, which we may assume is the sequence $A_3, A_4, \ldots$, such that each A_i is adjacent to A_{i+1}, with neither A' nor any other A_j between them. They must therefore be separated by a component M_i of $A \cap M$ that intersects both ends of A. The components M_i converge to a subcontinuum of A' that joins the ends of A. This shows that M is not locally connected at these points of A', a contradiction.

Suppose (1) is satisfied but (1′) is not. That is, there is a disk D in $\mathbf{S}^2$ and infinitely many large components of $D \cap M$. We may take a sequence of such components that converge to a nondegenerate subcontinuum of M. We take an annulus A that separates two points of the limit continuum. Infinitely many of the large components cross this annulus. They are separated by large components of $A \setminus M$ that cross the annulus. Arcs in these components that cross the annulus allow one to form a disk D that is crossed by infinitely many large components of $D \setminus M$, a contradiction to (1). We conclude that (1′) is satisfied.

Similar arguments show that (1′) implies (1) and that these are equivalent to (2) and (2′).

Finally, if M is not locally connected, then there is a point $p \in M$ and a neighborhood N of p in M such that p is a limit point of the components of $N \cap M$ that do not contain p. Each of these components intersects the boundary of M. These large components contradict (1′).

Theorem 2.4.2. Suppose that M is a Peano continuum in the 2-sphere $\mathbf{S}^2$, and suppose that U is a component of the complement of M in $\mathbf{S}^2$. Then there is a map $f : \mathbf{B}^2 \to \mathrm{cl}(U)$ from the 2-disk $\mathbf{B}^2$ onto the closure of the domain U that takes $\mathrm{int}(B^2)$ homeomorphically onto U and takes $\mathbf{S}^1 = \partial(\mathbf{B}^2)$ continuously onto $\partial(U)$. In addition, if A is a free boundary arc of $\mathrm{cl}(U)$, then we may assume that the map f is one to one over the arc A.

Remark. That the arc A is *free* means that A is accessible from precisely one of its sides from the domain U and that $\mathrm{int}(A)$ is an open subset of $\partial(U)$.

Indication of proof. There are well-known, completely topological proofs of this theorem. However, refinements of the Riemann Mapping Theorem also give very enlightening analytic information. The relevant analytic theory is the theory of *prime ends*. There is a good exposition of the theory in John B. Conway's readily available textbook, [4, Chapter 14, Sections 1-5]. It follows from the local connectivity of M (applying Theorem 2.4.1(1)) that the *impressions* of the prime ends in U are all singletons. By the theory of prime ends, the Riemann mapping from $\mathrm{int}(\mathbf{B}^2)$ onto U extends continuously to the boundary.

That the arc A is *free* means that A is accessible from precisely one of its sides from the domain U and that $\mathrm{int}(A)$ is an open subset of $\partial(U)$. Consequently, the prime ends at A correspond exactly to the points of A so that the map is one to one over A.

Proof of Theorem 1.2. Suppose that M is a locally connected continuum in $\mathbf{S}^2$. If $M = \mathbf{S}^2$, then M is already codiscrete. Otherwise, let $U_1, U_2, \ldots$ denote the complementary domains of M in $\mathbf{S}^2$. By Theorem 2.4.1, the components of $\mathbf{S}^2 \setminus M$ form a null sequence. By Theorem 2.4.2, there is for each i a continuous surjection $f_i : \mathbf{B}^2 \to \mathrm{cl}(U_i)$ that takes $\mathbf{S}^1$ onto the boundary of U_i and takes the interior of $\mathbf{B}^2$ homeomorphically onto U_i. Let $p_i = f_i(0)$. Then the set $D = \{p_1, p_2, \ldots\}$ is obviously discrete. The set

$\mathrm{cl}(U_i) \setminus \{p_i\}$ can obviously be deformed into the boundary of U_i by pushing points away from p_i along the images under f_i of radii in $\mathbf{B}^2$. These deformations can be combined to deform all of $X = \mathbf{S}^2 \setminus D$ onto M since the U_i form a null sequence. We conclude that M is homotopy equivalent to the codiscrete set $X = \mathbf{S}^2 \setminus D$.

Conversely, if X is codiscrete, then we may take, about the points p of $D(X)$, small disjoint round disks $d(p)$. The continuum $M = \mathbf{S}^2 \setminus \cup_p \mathrm{int}(d(p))$ is a Peano continuum to which X can be deformed by a strong deformation retraction.

This completes the proof of Theorem 1.2.

We may think of the proof of Characterization Theorem 1.1 as a substantial generalization of the proof of Theorem 2.4.2. We shall need an intermediate generalization of Theorem 2.4.2 that deals with compact sets that act much like Peano continua but are not necessarily connected. We shall deal with them by joining them together by arcs so as to form a Peano continuum.

Definition 2.5.1. A connected open subset U of $\mathbf{S}^2$ is called a *Peano domain* if its nondegenerate boundary components form a null sequence of Peano continua. [Note that there may be uncountably many additional components that are single points.]

Theorem 2.5.2. Suppose that U is a connected open subset of the 2-sphere $\mathbf{S}^2$. Then the following three conditions are equivalent:

(1) The open set U is a Peano domain.

(2) For each disk D in $\mathbf{S}^2$, the components of $U \cap D$ form a null sequence.

(3) There is a continuous surjection $f : \mathbf{B}^2 \to \mathrm{cl}(U)$ such that $f(\mathbf{S}^1) \supset \partial(U)$ and $f|\mathrm{int}(\mathbf{B}^2)$ is a homeomorphism onto its image.

Remark. Note that (1) generalizes the notion of local connectedness. Note that (2) generalizes characterization (1) of local connectedness in Theorem 2.4.1; the reader can reformulate (2) in each of the ways suggested by Theorem 2.4.1. Note that (3) generalizes Theorem 2.4.2. Note that, in the proof, we can assume that the map f is 1-1 over given free boundary arcs of U because the same thing is true in Theorem 2.4.2.

Proof. Assume (1), so that U is a Peano domain. Assume that (2) is not satisfied, so that there is a disk D in $\mathbf{S}^2$ such that the components of $U \cap D$ do not form a null sequence. Then some sequence $U_1, U_2, \ldots$ of components converges to a nondegenerate continuum M. The continuum M must be a subset of a boundary component of U. We may assume that the components $U_1, U_2, \ldots$ are separated from each other by large boundary components of U. There are only finitely many large boundary components of U. Hence infinitely many of the separators must come from the same boundary component. It follows that the limit, namely M, is also in the same boundary component. But this boundary component is not locally connected at the points of M, a contradiction. We conclude that (2) is satisfied so that (1) implies (2).

Assume that (2) is satisfied. Assume that (1) is not satisfied. Then either there is a component of $\partial(U)$ that is not locally connected, or there exist infinitely many components of $\partial(U)$ having diameter $\geq \epsilon$, for some fixed $\epsilon > 0$. In either case, taking a convergent sequence of large components, we find the existence of an annulus A in $\mathbf{S}^2$ and components $X_1, X_2, \ldots$ of $\partial(U) \cap A$, each of which intersects both components of $\partial(A)$. These components $\partial(U) \cap A$ must be separated by large components of $A \cap U$. if we remove a slice from one of these large separating components, we obtain a disk D that is crossed by infinitely many large components of $U \cap D$, which contradicts (2). Therefore (2) implies (1).

Assume that (3) is satisfied, so that there is a continuous surjection $f : \mathbf{B}^2 \to \mathrm{cl}(U)$ such that $f(\mathbf{S}^1) \supset \partial(U)$ and $f|\mathrm{int}(\mathbf{B}^2)$ is a homeomorphism onto its image. Assume that (1) is not satisfied, so that there is either a component of $\partial(U)$ that is not locally connected, or, there exist infinitely many components of $\partial(U)$ each having diameter greater than some fixed positive number ϵ. In either case, we find by taking limits that there is an annulus A in $\mathbf{S}^2$ and components $X_1, X_2, \ldots$ of $\partial(U) \cap A$, each of which intersects both components of $\partial(A)$. We may assume that $X_1, X_2, \ldots$ converges to a continuum X_0 joining both components of $\partial(A)$. We may assume that $X_{i-1} \cup X_{i+1}$ separates X_i from X_0 in A, for $i = 2, 3, \ldots$.

Pick $p_i \in X_i \cap \text{int}(A)$ such that $p_1, p_2, \ldots \to p_0$. Let $q_0, q_1, q_2, \ldots \in \mathbf{S}^1$ be points such that $f(q_i) = p_i$. Let B_i be the straight-line segment in $\mathbf{B}^2$ joining q_0 to q_i. We may assume that the arcs B_i converge to an arc or point B in $\mathbf{B}^2$. We shall obtain a contradiction as follows.

The image $f(B_i)$ joins X_i to X_0. It misses $X_{i-1} \cup X_{i+1} \subset \partial(U)$ since $f(q_i) \in X_i$, $f(q_0) \in X_0$, and $f(\text{int}(B_i) \subset U$. Hence, traversing B_i from q_i toward q_0, there exists a first point $b_i \in B_i$ such that $f(b_i) \in \partial(A)$. We may assume that $b_i \to b \in \mathbf{B}^2$ and $f(b_i) \to f(b_0) \in \partial(A)$. Since $f(b_i)$ is separated from X_0 by $X_{i-1} \cup X_{i+1}$ in A and since $X_i \to X_0$, we may conclude that $f(b_0) \in X \cap \partial(A)$. Hence $b_0 \in \mathbf{S}^1 \setminus \{q_0\}$. But b_0 must therefore be an endpoint of B distinct from q_0 and must therefore be the limit of the points q_i. We find that $f(q_i) \to p_0 \in \text{int}(A)$ and $f(q_i) \to f(p_0) \in \partial(A)$, a contradiction.

We conclude that (3) implies (1).

It remains to prove that (1) implies (3). This is by far the hardest of the implications. It is a generalization of the rather deep Theorem 2.4.2, and we shall reduce it to that theorem. We shall also make use of the wonderful R.L. Moore Decomposition Theorem 2.3.1.

Our plan is to connect $\partial(U)$ by deleting from U a null sequence $A_1, A_2, \ldots$ of arcs to form a new connected open set $V = U \setminus \cup_i A_i$ whose boundary $\partial(V) = \partial(U) \cup \bigcup_i A_i$ is a locally connected continuum. Then we simply apply Theorem 2.4.2.

For convenience, we smooth the nondegenerate components C of $\partial(U)$ as follows. We define U_C to be the component of $\mathbf{S}^2 \setminus C$ that contains U. Since C is locally connected by (1), we may apply Theorem 2.4.2 to find a continuous surjection $g : \mathbf{B}^2 \to C \cup U_C$ that takes $\mathbf{S}^1$ onto C and takes $\text{int}(\mathbf{B}^2)$ homeomorphically onto U_C. Thus, pulling U_C radially into itself along the images of radii, we find that we lose no generality in assuming that C is a topological circle. Since the nondegenerate components of $\partial(U)$ form a null sequence by (1), we may repeat the argument infinitely often to conclude that we lose no generality in assuming that each nondegenerate component is a simple closed curve. That is, U is the complement of a null sequence of disks $D_1, D_2, \ldots$ and a 0-dimensional set D, the union of $D_1, D_2, \ldots$, and D being closed.

We wish to construct a nice sequence of cellulations of the 2-sphere that respect the boundary components of U. If, for example, we wish to concentrate on some particular finite set S of the large disks D_i, we may form an upper semicontinuous decomposition of $\mathbf{S}^2$ by declaring the other D_i's that miss S to be the nondegenerate elements of the decomposition. By R.L. Moore's Decomposition Theorem 2.3.1, the quotient space is the 2-sphere $\mathbf{S}^2$. The (homeomorphic) image of U in this new copy of $\mathbf{S}^2$ will have, as complement, the (images of the) elements of S and a 0-dimensional set that is closed away from S. It is then an easy matter to cellulate $\mathbf{S}^2$ so that the elements of S cover a subcomplex and the remainder of the 1-skeleton misses ∂U entirely.

As a consequence, we find that there is a sequence $S_1, S_2, \ldots$ of arbitrarily fine cellulations of $\mathbf{S}^2$, S_{i+1} subdividing S_i, such that, for each i, the following conditions are satisfied:

(i) Two 2-cells of S_i that intersect intersect in an arc.

(ii) The 1-skeleton of S_i misses all of the 0-dimensional part D of $\partial(U)$.

(iii) $\forall j$, the 1-skeleton of S_i either misses the disk D_j or contains $\partial(D_j)$. Consequently, S_i has a distinguished finite subcollection of disks D_j that are precisely equal to unions of 2-cells of S_i. All other disks D_k will lie in the interiors of 2-cells of S_i.

(iv) If a 2-cell C of S_i has a boundary point in some $\partial(D_j)$, with $\text{int}(C) \not\subset D_j$, then $\partial(C) \cap \bigcup_k D_k$ is an arc in $\partial(D_j)$.

We shall string the components of ∂U together by arcs that run through U. These arcs will be built by approximation. The ith approximation will consist of arcs that join certain 2-cells of the cellulation S_i.

It is necessary to distinguish four types of 2-cells in the cellulation S_i:

A 2-cell C of S_i is of type 0 if it lies entirely in U.

A 2-cell C is of type 1 if it lies entirely in the complement of U, hence lies in one of the distinguished disks D_j of the cellulation S_i (see (iii) above).

A 2-cell C is of type 2 if it intersects both U and the complement of U, but its boundary lies entirely in

U.

A 2-cell C is of type 3 if its boundary intersects both U and the complement of U. Condition (iv) above implies that a 2-cell C of type 3 has boundary that intersects precisely one disk D_j, that D_j is one of the distinguished disks of S_i, and the intersection is a boundary arc of each.

We shall essentially ignore the 2-cells of type 0. We shall deal with the disks of type 1 only implicitly by considering instead their unions that give the distinguished disks D_j of the cellulation S_i (see (iii) above). Cells of type 2 will be joined to these distinguished disks by arcs in U. Cells of type 3 will be joined to these distinguished disks by their intersecting boundary arcs.

It will be convenient to use the notation C^* for the union of the elements of a collection C of sets.

Let $\mathcal{D}_1$ denote the collection of D_j's that are distinguished in the cellulation S_1. Then $\mathcal{D}_1^* = \cup\{D \in \mathcal{D}_1\}$. We may assume $D_1 \in \mathcal{D}_1$. We may pick a collection of arcs $\mathcal{A}_1$ from the 1-skeleton $S_1^{(1)}$ of S_1 that irreducibly joins together these distinguished disks $D_j \in \mathcal{D}_1$ and the cells of S_1 of type 2. Then $\mathcal{C}_1 = \mathcal{D}_1^* \cup \mathcal{A}_1^*$ is a contractible set.

All of the cells of S_1 of type 1 are contained in $\mathcal{C}_1$. We may consider all cells of type 2 and 3 as attached to this contractible set in the following way. For each cell C of type 2, pick one point of intersection with $\mathcal{C}_1$ as attaching point. For each cell C of type 3, pick as attaching arc the boundary arc of C that lies in a distinguished disk.

We proceed by induction. We assume that we have constructed contractible sets $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \mathcal{C}_i$, that lie except for distinguished disks of $S_1, S_2, \ldots, S_i$, in the 1-skeletons of the cellulations. We may impose one additional condition on the cellulation S_{i+1}:

(v) For each cell C of S_i that has type 2 or 3, that part of the 1-skeleton of S_{i+1} that lies in the interior of C, taken together with the attaching point (type 2) or attaching arc (type 3), is connected.

All of the action in creating $\mathcal{C}_{i+1}$ takes place in the individual cells C of S_i of type 2 and 3. We may pick a collection of arcs $\mathcal{A}_{i+1}(C)$ from that part of the 1-skeleton of S_{i+1} that lies in the interior of C, taken together with the attaching point (type 2) or attaching arc (type 3), that irreducibly joins together the attaching set of C, the distinguished disks $D_j \in \mathcal{D}_{i+1}$ in C, and the cells of S_{i+1} of type 2 in C. All of these new distinguished disks and all of these new arcs can be added to $\mathcal{C}_i$ to form a new contractible set $\mathcal{C}_{i+1}$. We denote the entire union $\bigcup_C \mathcal{A}_{i+1}(C)$ of arcs as $\mathcal{A}_{i+1}$.

For each of the new cells of types 2 and 3, we choose an attaching point or arc as before.

We leave it to the reader to verify that $M = (\mathbf{S}^2 \setminus U) \cup \bigcup_i (A_i)$ is a single locally connected continuum with a single complementary domain $V = U \setminus \bigcup_i (A_i)$.

By Theorem 2.4.2, there is a map $f : \mathbf{B}^2 \to \mathrm{cl}(V)$ from the 2-disk $\mathbf{B}^2$ onto the closure of the domain V that takes $\mathrm{int}(B^2)$ homeomorphically onto V and takes $\mathbf{S}^1 = \partial(\mathbf{B}^2)$ continuously onto $\partial(V)$. The same map establishes condition (3) of Theorem 2.5.2.

This completes the proof that (1) implies (3). Thus all three conditions of Theorem 2.5.2 are equivalent, as claimed. The proof of Theorem 2.5.2 is therefore complete.

Our final theorem of this section shows how to push a Peano domain onto its boundary together with a 1-dimensional set provided the domain is punctured on a nonempty discrete set. This easy theorem will be needed as the last step in the proof of Theorem 1.1.

Theorem 2.5.3. Suppose that U is a Peano domain in $\mathbf{S}^2$ and that C is a nonempty countable or finite subset of U that has no limit points in U. Then $\mathrm{cl}(U) \setminus C$ can be retracted by a strong deformation retraction onto a 1-dimensional set that contains $\partial(U)$.

Proof. By Theorem 2.5.2, we know that there is a continuous surjection $f : \mathbf{B}^2 \to \mathrm{cl}(U)$ such that $f(\mathbf{S}^1) \supset \partial(U)$ and $f|\mathrm{int}(\mathbf{B}^2)$ is a homeomorphism onto its image.

Since $f(\mathrm{int}(\mathbf{B}^2))$ is dense in $f(\mathbf{B}^2) = \mathrm{cl}(U)$ and disjoint from $f(\mathbf{S}^1)$, $f(\mathbf{S}^1)$ must be 1-dimensional. Hence it is an easy exercise to show that we may modify f slightly over U so that $f(\mathbf{S}^1)$ misses C. We may further modify f so that f maps the origin $0 \in \mathbf{B}^2$ to a point of C and so that all other points of C have preimages on different radii of $\mathbf{B}^2$. Let $f^{-1}(C) = \{c_0 = 0, c_1, c_2, c_3, \ldots\}$. Let $A_1, A_2, \ldots$ be the radial arcs

beginning at $c_1, c_2, \ldots$, respectively, and ending on $\mathbf{S}^1 = \partial(\mathbf{B}^2)$. Let $D_1, D_2, \ldots$ be disjoint round disks in $\mathrm{int}(\mathbf{B}^2) \setminus \{0\}$ centered at $c_1, c_2, \ldots$, respectively, such that the only A_j intersected by D_i is A_i. Let $V = \mathrm{int}(\mathbf{B}^2) \setminus [\bigcup_i A_i \cup \bigcup_i D_i]$. Then $\mathbf{B}^2 \setminus f^{-1}(C)$ can obviously be retracted by a strong deformation retraction onto the 1-dimensional set $\partial(V)$. Hence $f(\mathbf{B}^2) \setminus C = \mathrm{cl}(U) \setminus C$ can be retracted by a strong deformation retraction onto the 1-dimensional set $f(\partial(V))$.

3. The necessity of conditions (1) and (2) in Theorem 1.1.

We assume that X is a codiscrete set that is homotopy equivalent to a metric 1-dimensional set Y. Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverses.

We isolate the three key technical constructions as lemmas. Each of these is standard and well-known. We omit the proofs.

Dimension Lemma 3.1. If $g : Z' \to Z$ is any map from a 1-dimensional compactum Z' into the closure Z of an open subset U of $\mathbf{S}^2$, then g is homotopic, by a homotopy which only moves points in U to a map $g' : Z' \to Z$ such that $g'(Z') \cap U$ is 1-dimensional. [The key ideas are explained, for example, in [12, Exercises for Chapter 3, Sections G and H].]

Homotopy Lemma 3.2. (i) Let $C \subset \mathbf{S}^2$ be closed, and let $H : C \times [0,1] \to \mathbf{S}^2$ denote a deformation of C that begins at the identity (that is, $\forall c \in C$, $H(c, 0) = c$). Then H can be extended to a deformation $H' : \mathbf{S}^2 \times [0,1] \to \mathbf{S}^2$. (ii) If H moves no point as far as $\epsilon > 0$, then we may require that H' have the same property. (iii) If N is an open set containing the support of $H|\partial C$ in $\mathbf{S}^2$, then we may require that N contain the support of $H'|\mathbf{S}^2 \setminus C$. [See [, **S**62, Lemma 62.1 and Exercise 3.]

Ring Lemma 3.3. Suppose condition (2) of Theorem 1.1 fails. Then there are a ring R' in $\mathbf{S}^2$ and components $U_1', U_2', \ldots$ of $R' \setminus B(X)$ such that each U_j' intersects both boundary components of R' and misses the set $D(X)$. [See Theorem 2.4.1 and its proof.]

The three lemmas imply the theorem as follows: By precomposing the homotopy equivalence f with a deformation retraction onto a compact subset of X, we may assume that the image $f(X)$ is a 1-dimensional continuum Z'. By Dimension Lemma 3.1, we may assume that $g \circ f(X) \setminus B(X)$ is 1-dimensional. Let $G : X \times [0,1] \to \mathbf{S}^2$ be a homotopy that begins with the identity on X and ends with $g \circ f$. By Theorem 2.1, we see that $G(x, t) = x$ for each $x \in B(X)$.

Assume that condition (1) of the hypothesis of Theorem 1.1 fails, so that there is some component U of $\mathbf{S}^2 \setminus B(X)$ contains no point of $D(X)$. Hence $U \subset X$. Let $H : \mathrm{cl}(U) \times [0,1] \to \mathbf{S}^2$ denote the restriction of G to $\mathrm{cl}(U) \times [0,1]$. Since H fixes $\partial U \subset B(X)$, we may extend H to a deformation H' of $\mathbf{S}^2$ that fixes $\mathbf{S}^2 \setminus U$ pointwise. Since $H'(\mathbf{S}^2 \times \{1\}) \cap U \subset G(\mathbf{S}^2 \times \{1\}) \cap U$ is 1-dimensional, we see that H' deforms $\mathbf{S}^2$ into a proper subset of itself, which is impossible. Hence condition (1) must be satisfied.

Assume that condition (2) of the hypothesis of Theorem 1.1 fails. Then, by Ring Lemma 3.3, there are a ring R' in $\mathbf{S}^2$ and components $U_1', U_2', \ldots$ of $R' \setminus B(X)$ such that each U_j' intersects both boundary components of R' and fails to intersect the set $D(X)$.

By passing to a subsequence, we may assume that the components $U_1', U_2', \ldots$ converge to a continuum A that joins the two boundary components of R'. Since the components U_j' are separated by $B(X)$, it follows that $A \subset B(X)$. Let D be a small disk in $\mathrm{int}(R')$ centered at some point of A. Since the deformation G constructed above moves no point of $B(X)$, there is a neighborhood N of A in X, no point of which is moved by G as far as $1/2$ the distance from ∂R to D. We shoose j so large that $\mathrm{cl}(U_j) \subset N$ and $U_j \cap \mathrm{int}(D) \neq \emptyset$. Since no point of $D(X)$ lies in U_j, all of $\mathrm{cl}(U_j)$ lies in X.

We let $H : \mathrm{cl}(U_j) \times [0,1] \to \mathbf{S}^2$ be the restriction of G to $\mathrm{cl}(U_j) \times [0,1]$. By Homotopy Lemma 3.2(i), there is a deformation $H' : \mathbf{S}^2 \times [0,1] \to \mathbf{S}^2$ that extends H. By Homotopy Lemma 3.2(iii), we may require that $H'|[\mathbf{S}^2 \setminus \mathrm{cl}(U_j)] \times [0,1]$ move points only near $(\partial R) \cap \mathrm{cl}(U_j)$, a set that contains the support of $H|\partial U_j \times [0,1]$. By Homotopy Lemma 3.2(iii), we may require that no points of $\mathbf{S}^2 \setminus U_j$ be carried into $D \cap U_j$. Hence H' is a homotopy of $\mathbf{S}^2$ that takes $\mathbf{S}^2$ to a proper subset of itself, an impossibility. Hence condition (2) of Theorem 1.1 is also satisfied.

4. The sufficiency of conditions (1) and (2) in Characterization Theorem 1.1.

We assume conditions (1) and (2) of Characterization Theorem 1.1. That is, the open set $U_0 = \mathbf{S}^2 \setminus B(X)$

satisfies the following two conditions:

(1) Each component of U_0 contains a point of $D(X)$.

(2) If D is any disk in $\mathbf{S}^2$, then the components of $U_0 \cap D$ that contain no point of $D(X)$ form a null sequence.

Our goal is to show that X is homotopy equivalent to a 1-dimensional set.

Notice that properties (1) and (2) make no explicit mention of the bad set $B(X)$ and are simply properties that an open subset of $\mathbf{S}^2$ may or may not have. This is an important observation, because our proof that X is homotopy equivalent to a 1-dimensional set will involve a complicated induction that will involve a decreasing sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$ of open sets, each of which satisfies properties (1) and (2).

It will also be convenient to adopt the following terminology: we say that set is *punctured* if it contains a point of $D(X)$. Otherwise, we say that it is *unpunctured*.

We first have to deal with the trivial case where $B(X) = \emptyset$. If $B(X) = \emptyset$, then the single component $\mathbf{S}^2 = \mathbf{S}^2 \setminus B(X)$ must contain a point of $D(X)$ by (1). Thus there must be at least one point of $D(X)$ and at most finitely many. Hence X is clearly homotopy equivalent to a point or bouquet of circles.

From now on, we may assume that the set $D(X)$ is infinite and the set $B(X)$ is nonempty. Since $D(X)$ is countable, we may list the points $p_0, p_1, p_2, \ldots$ of $D(X)$. We need to show that X is homotopy equivalent to a 1-dimensional set. We shall do this by constructing a null sequence $U_0, U_1, U_2, \ldots$ of disjoint Peano domains such that, for each i, $p_i \in U_i$, and such that the union $\cup_i U_i$ is dense in $\mathbf{S}^2$. Each set $\mathrm{cl}(U_i) \setminus \{p_i\}$ can be deformed onto a 1-dimensional set that contains its boundary by Theorem 2.5.3. Since these sets form a null sequence, the deformations can be combined to find a deformation that takes X onto the union of $\mathbf{S}^2 \setminus \cup_i U_i$ and the sets $\partial(U_i)$. Each of these sets is a compact 1-dimensional set. Hence their (countable) union is 1-dimensional.

The domains U_i are created by a long induction. Each step of the induction constructs a null sequence of Peano domains. At step 0 of the induction, an individual domain can have diameter as large as the diameter of $\mathbf{S}^2$. Thereafter, however, we may restrict the maximum diameter of a Peano domain at step i to be bounded by $1/i$. Hence the union of this countable collection of null sequences is also a null sequence.

We consider $\mathbf{S}^2$ as $\mathbf{R}^2 \cup \{\infty\}$. We may assume that $p_0 = \infty \in D(X)$. By scaling and translating $\mathbf{R}^2$, we find that we may assume that $[D(X) \setminus \{\infty\}] \cup B(X)$ lies in the interior of the closed unit square $S = [0,1] \times [0,1]$.

We begin now the construction of our first null sequence of Peano domains. We outline the strategy. The reader who digests this strategy will be able to avoid getting lost in the details. We are trying to fill the open set $U_0 = \mathbf{S}^2 \setminus B(X)$ with small Peano domains, more precisely a null sequence of Peano domains, that are punctured (contain points of $D(X)$). We therefore cover U_0 with a fine grid to divide it into small pieces. What happens then is reminiscent of the children's story, "Fortunately, Unfortunately." Fortunately, some of these small pieces will be punctured. Unfortunately some will be unpunctured. Fortunately, the unpunctured pieces form a null sequence by hypothesis (2); unfortunately, however, they must be attached to adjacent pieces that are punctured and, unfortunately, the adjacent punctured pieces need not form a null sequence. Fortunately, we can carve out of the adjacent punctured pieces a null sequence of smaller punctured pieces to which we can attach the unpunctured pieces. Unfortunately, the process of carving out small punctured pieces creates new unpunctured pieces. Fortunately, the new unpunctured pieces form a null sequence that we can attach to the null sequence of punctured pieces. Unfortunately, the carving out of small punctured pieces creates new, as yet unattached, punctured pieces that need not form a null sequence. Fortunately, the unattached punctured pieces are uniformly small and, together, form a new open set U_1 that satisfies hypotheses (1) and (2). We can then undertake the inductive step with a new open set whose pieces are smaller than at the previous stage. Here are the details.

Step 1. Creating small pieces. We impose a square grid on S consisting of a large square formed from small constituent closed squares. Since the set $D(X)$ is countable, we lose no generality in assuming that the edges of the grid miss $D(X)$. The grid divides the open set $U_0 = \mathbf{S}^2 \setminus B(X)$ into many components. We call the collection of such components $\mathcal{C}_0$. More precisely: **(i)** The set $\mathbf{S}^2 \setminus \mathrm{int}(S)$ is an element of $\mathcal{C}_0$; **(ii)** If T is any small, closed, constituent square of the grid, then each component of $T \setminus B(X)$ is also an element

of $\mathcal{C}_0$. Note that the elements of $\mathcal{C}_0$ are not in general disjoint since they can intersect along the edges of the grid.

Step 2. Collecting the unpunctured pieces into a null sequence of small sets. Let $\mathcal{C}_0'$ denote the subcollection of $\mathcal{C}_0$ consisting of those elements whose interiors are unpunctured. We take the union $\cup\mathcal{C}_0'$ of the elements of $C_)'$ and claim two things: **(iii)** The components of $\cup\mathcal{C}_0'$ form a null sequence, and **(iv)** Each component of $\cup\mathcal{C}_0'$ shares an edge with an element of $\mathcal{C}_0$ whose interior is punctured.

Proof of (iii). We apply here the fundamental principle of convergence of continua from Section 2.2. The argument could be repeated almost verbatim perhaps four more times in the course of Section 2. Often we will have to consider two cases, depending on whether the limit continuum contains a point in the interior of a constituent square of the superimposed grid or does not. We will not always repeat the details after this first argument. Here are the details:

Suppose $\epsilon > 0$, and suppose that there exist components $Y_1, Y_2, \ldots$, each of diameter $\geq \epsilon$. We may assume that $Y_i \to Y$ in the sense of Section 2.2, where Y is a continuum of diameter $\geq \epsilon$.

Suppose first that Y contains a point in the interior of some constituent square. Then a small annulus A about that point in the interior of the constituent square intersects all but finitely many of the Y_i in a component that crosses A from one boundary component to the other, which easily gives a contradiction to hypothesis (2).

Suppose next that Y lies in the 1-skeleton of the grid. Then it contains an interval of an edge of one of the small constituent squares. In this case, we may take an annulus A that surrounds an interior point of the interval and intersects each of the two adjacent squares in a disk (half of an annulus). Again, all but finitely many of the Y_i will intersect one of these two disks in a component that crosses the disk from one side to the opposite, which easily gives a contradiction to hypothesis (2).

This completes the proof of (iii).

Proof of (iv). We may expand the elements of $\mathcal{C}_0$ slightly without introducing intersections between sets that did not already intersect; we obtain thus an open covering of U_0. Each component of U_0 is punctured, by hypothesis (1). In each component V, any two elements of $\mathcal{C}_0$, as expanded, that lie in V are joined by a finite chain of such elements by a standard connectedness argument. A minimal such chain connects each element of $\mathcal{C}_0'$ to an element of $\mathcal{C}_0$ that is punctured. Property (iv) follows.

Step 3. Attaching the unpunctured pieces of Step 2 to a null sequence of punctured pieces. To each component K of $\cup\mathcal{C}_0'$ we assign a punctured element $L = L(K) \in \mathcal{C}_0$ that intersects K along at least one edge. Such an element $L(K)$ exists by (iv) of Step 2. The elements L thus chosen definitely need not form a null sequence, but we shall carve out from such elements L a new null sequence of punctured domains to which we may attach the components K. Here is the argument:

For each component K, choose an open arc $A(K)$ along which K is attached to $L(K)$. Choose a point $p(K) \in A(K)$. Enumerate these points as $q_1, q_2, \ldots$. Each q_i belongs to a specific K_i, and arc A_i, and component $L_i = L(K_i)$.

Choose an arc B_1 in L_1 that joins q_1 to $D(X)$ irreducibly. We may require that $B_1 \cap(\text{1-skeleton of grid}) = q_1$ and that, $\forall$ arcs B having the same properties, $\text{diam}(B_1) \leq 2\text{diam}(B)$.

Proceed inductively. Choose an arc B_{k+1} in L_{k+1} joining q_{k+1} to $D(X) \cup B_1 \cup \cdots \cup B_k$ irreducibly. We may require that $B_{k+1} \cap (\text{1-skeleton of grid}) = q_{k+1}$ and that, $\forall$ arcs B having the same properties $\text{diam}(B_{k+1}) \leq 2\text{diam}(B)$.

We make the following claims about the arcs B_i:

(v) The arcs $B_1, B_2, \ldots$ form a null sequence.

(vi) $\forall \epsilon > 0$, $\exists k$ such that each component of $\mathbf{B}(k) = B_{k+1} \cup B_{k+2} \cup \cdots$ has diameter less than ϵ.

[Note that (vi) implies (v). Properties (v) and (vi) are stated separately since (v) is used in the proof of (vi).]

Proof of (v). Suppose that (v) is not satisfied. Then there is a subsequence $B_{i_1}, B_{i_2}, \ldots$ that converges to a nondegenerate continuum B. [This is our second application of the fundamental principal of Section 2.2.]

We may assume that the B_{i_j} all lie in the same small constituent square T of the grid and that their initial endpoints $q_{i_1}, q_{i_2}, \ldots$ converge to a point $q \in \partial T$. Let A be a small annulus about q that intersects T in a small disk A'. All but finitely many of the arcs B_{i_j} cross that disk A' in a large component B'_{i_j}. By hypothesis (2), only finitely many components of $A' \cap U_0$ do not contain a point of $D(X)$. It follows easily that either some B'_{i_j} is in a component that contains a point of $D(X)$ or is in a component that contains another B'_{i_k}, with $j > k$. In either case, the diameter of B_{i_j} can be considerably reduced by shortcutting B_{i_j} to $D(X)$ or to B_{i_k}, a contradiction. This completes the proof of (v).

Proof of (vi). We shall make strong use of (v). suppose there is an $\epsilon > 0$ such that each of the sets $\mathbf{B}(k) = B_{k+1} \cup B_{k+2} \cup \cdots$ contains a component Y_k of diameter $\geq \epsilon$. We may pick from Y_k a subset Y'_k that is a finite chain of the arcs $B_1, B_2, \ldots$ and that has diameter $\geq \epsilon$. Passing to a subsequence if necessary, we may assume that the sets Y'_k are disjoint and that all lie in the same small constituent square T. If $Y'_k = B_{k_1} \cup \cdots B_{k_\ell}$, with $k_1 < \cdots < k_\ell$, then we call $q(k) = q_{k_\ell}$ the initial point of Y'_k. We may assume that the initial points $q(k)$ converge to a point $q \in \partial(T)$. Let A be a small annulus about q that intersects T in a small disk A'. Then each Y'_k is a chain of small arcs crossing A' whose links B_{k_j} all intersect $\partial(T)$. There can be at most two such that are disjoint, a contradiction. This completes the proof of (vi).

From property (vi) it follows easily that each component B of $B_1 \cup B_2 \cup \cdots$ is a tree that lies in a single small constituent square T, contains exactly one point of $D(X)$, and has, as its leaves, special attaching points q_j in corresponding attaching arcs A_j of certain components K_j of $\cup\mathcal{C}'_0$. Furthermore, these trees B form a null sequence of trees. Each component of $\cup\mathcal{C}'_0$ is attached to one of these trees at a leaf. We thicken each of these trees slightly and disjointly so that they still form a null sequence, still contain one point of $D(X)$ each, but now intersect the appropriate attaching arcs A_j in neighborhoods A'_j of the attaching points q_j. The interiors of the thickened trees B' are clearly Peano domains since it is an easy matter to construct a continuous surjection $f : \mathbf{B}^2 \to \mathrm{cl}(B')$ that takes $\mathrm{int}(\mathbf{B}^2)$ homeomorphically onto $\mathrm{int}(B')$. These Peano domains will form the cores of the Peano domains that we are attempting to construct in this stage of the induction. To them, we must attach the components K_j that we have described above and also certain sets that we will describe in the next step.

Step 4. Attaching the unpunctured components created by removing the thickened trees of Step 3. When we remove the thickened trees B'_i from the components $L = L(K)$, we may create new components that are unpunctured. We must attach each of those to an adjacent thickened tree B'_i. The following property establishes the fact that these new unpunctured domains form a null sequence.

(vii) Let $\mathcal{C}''_0$ be the collection of sets defined as follows. If K is a component of $\cup\mathcal{C}'_0$ and $L = L(K) \in \mathcal{C}_0$, then the components M of $L \setminus \cup_{i=1}^{\infty} B'_i$ that contain no points of $\mathcal{C}''_0$ form a null sequence.

Proof of (vii). Suppose not. Then there are components $M_1, M_2, \ldots$ converging to a nondegenerate continuum M.

Suppose first that M has a point p that lies in the interior of a small constituent square T. Since $\cup_i B_i$ is locally a finite graph away from the edges of the grid, and a finite graph separates an open set locally into only finitely many components, $p \notin \cup_i B_i$. Hence there is a small annulus A surrounding p that contains no point of $\cup_i B_i$. Each M_i crosses A in a "large" set, contained in a component of $A \cap U_0$ that contains no points of $D(X)$ and no points of $\cup_i B_i$. There are only finitely many such, a contradiction.

Suppose finally that M lies in the 1-skeleton of the grid. Then we may suppose that M contains a nondegenerate interval I of an edge of a small donstituent square T, and we may assume that each M_i also lies in that square. We may take a small disk neighborhood A of $I' \subset I$ in T so that all but finitely many M_i cross A from one side to the other near I'. No point of $\mathrm{int}(I)$ can lie in $\cup_i B_i$, for $\cup_i B_i$ separates into only two components near such a point. Hence, only large B_i's can be near I'. Hence I' has a neighborhood in A missing $\cup_B i$. But, by hypothesis (2), all but finitely many of the components crossing A must contain points of $D(X)$, a contradiction.

This completes the proof of (vii).

Each of the components M just discussed share an arc with some thickened tree B'. We attach each component M to such an adjacent B' along an attaching arc.

Step 5. Completion of the first null sequence of Peano domains. We have at this point created

three null sequences of sets, namely, the components K of $\cup\mathcal{C}_0'$, the components B' of thickened trees, and the unpunctured components M that were formed when the thickened trees were carved out of punctured components of $\mathcal{C}_0$. Using the attaching arcs described earlier, we can therefore form a null sequence of domains of the form $V = \mathrm{int}(B' \cup K_1 \cup K_2 \cup \cdots \cup M_1 \cup M_2 \cup \cdots)$, where B' is a thickened tree and the K's and the M's are attached to B' along attaching arcs.

(viii) The sets V, which obviously form a null sequence of sets, are all Peano domains.

Proof of (viii). We have already noted that $\mathrm{int}(B')$ is a Peano domain. Each of the sets K_i is a Peano domain because, by hypothesis (2) of this theorem, it satisfies hypothesis (2) of Theorem 2.5.2. We see that the sets M_j are Peano domains because of the following argument. Suppose there is a disk D such that the components of $M_j \cap D$ do not form a null sequence. We let $V_1, V_2, \ldots$ denote a sequence of components converging to a nondegenerate continuum V. We get a contradiction exactly as in the argument for (vii) above.

We now choose, for the closures of B' and for the closures of each of the K_i's and each of the M_j's a continuous surjection from $\mathbf{B}^2$ as in condition (3) of Theorem 2.5.2. By the proof of Theorem 2.5.2, as noted in the remark following the statement of Theorem 2.5.2, we may assume that these maps are 1-1 over the attaching arcs. It is thus an easy matter to piece these functions together to get a single continuous surjection from $\mathbf{B}^2$ onto the closure of $V = \mathrm{int}(B' \cup K_1 \cup K_2 \cup \cdots \cup M_1 \cup M_2 \cup \cdots)$ of the kind required by Theorem 2.5.2(3).

This completes the proof of (viii).

Step 6. Preparing for the next stage of the induction. If K is an element of $\mathcal{C}_0$ from which certain thickened trees B_i' have been removed, then the remaining punctured components all have diameter less than or equal to the mesh of the covering grid. However, they need not form a null sequence. We simply take the union of the interiors of such elements in $\mathbf{R}^2$ to form a new open set U_1. This open set forms the input to the next stage of the induction. We need to verify the following fact:

(ix) The open set U_1 satisfies the two conditions (1) and (2) with which we began Section 4.

Proof of (ix). The remaining components are all subsets of components of elements of $\mathcal{C}_0$, hence have diameter less than or equal to the mesh of the covering grid.

Suppose that D is a disk and $D \cap U_1$ has infinitely many large components M_i that contain no point of $D(X)$. We may assume $M_i \to M$, M nondegenerate. We argue again exactly as in the proof of (vii) to obtain a contradiction.

Thus hypothesis (2) is satisfied. Since each component of U_1 is, by hypothesis, punctured, hypothesis (1) is also satisfied.

This completes the proof of (ix).

Step 7. The inductive step and the completion of the proof. We now recycle the new open set U_1 as the set U_0 of the argument just given, but use a grid with much smaller mesh. We repeat this process inductively, infinitely often. The completion of the argument is then clear provided we make the following two remarks:

(x) We may require that the point $p_i \in D(X)$ lie in one of the trees in the ith stage of the induction.

Indeed, we may choose the mesh so small that, if p_i has not been used before stage i, then p_i is the only point of $D(X)$ in a square of the grid and its neighboring squares, all lying in U_i. We can choose to attach the neighboring squares to the square containing p_i.

(xi) Eventually, every point p of $\mathbf{S}^2 \setminus (D(X) \cup B(X))$ lies in the closure of the constructed Peano domains.

Indeed, when squares are sufficiently small, every square containing p misses $D(X) \cup B(X)$. If p has not already appeared in the closure of one of our Peano domains, then p will lie in a component K that contains no point of $D(X)$, hence will be attached to some thickened tree at that stage.

Thus our proof is complete that we can tile the complement of $B(X)$ with a null sequence of disjoint Peano domains. Hence, infinitely many applications of Theorem 2.5.3 show that X can be deformed by a strong deformation retraction onto a 1-dimensional set.

5. Proof of Theorem 1.4. We are given a codiscrete set X. By Theorem 1.2, X is homotopy equivalent to a planar Peano continuum M. We work with M. We must show that the fundamental group of M embeds in the fundamental group of a 1-dimensional planar Peano continuum M'.

The construction of the 1-dimensional planar Peano continuum M'. We shall associate with M a quotient map $\pi : M \to M'$ onto a 1-dimensional Peano continuum M' in such a way that each nondegenerate point preimage $\pi^{-1}(x)$, for $x \in M'$, is an arc in M with endpoints in ∂M.

Adjusting M. After slight adjustment, we may assume that ∂M contains no vertical interval.

Proof. Suppose $\epsilon > 0$ given. It suffices to show how to move M a distance $< \epsilon$ such that no homeomorphic copy of M near the new M can have boundary that contains a vertical interval as large as ϵ.

Let $[a, b] \times [c, d]$ be a rectangle that contains the 2ϵ neighborhood of M, with the interval $[a, b]$ horizontal and the interval $[c, d]$ vertical. Let $c = y_0 < y_1 < \cdots < y_k = d$ be a partition of $[c, d]$ such that $y_i - y_{i-1} < \epsilon/2$ for each i. Let $[a_j, b_j] \times \{Y_j\}$, $j = 1, \ldots, \ell$, be a collection of horizontal intervals of length $< \epsilon/2$, all levels $Y_1, \ldots, Y_\ell, y_0, \ldots, y_k$ being distinct, such that, if v is a vertical interval that joins adjacent intervals $[a, b]\times\{y_i\}$, then v intersects some $[a_j, b_j] \times \{Y_j\}$.

Since ∂M is nowhere dense in $\mathbf{R}^2$, there exists a small open ball B_j in $\mathbf{R}^2 \setminus \partial M$ arbitrarily close to $[a_j, b_j]\times\{Y_j\}$. If such a ball is properly chosen, it is possible to expand all of the B_j's by an ϵ-homeomorphism of $\mathbf{R}^2$ so that the image of B_j contains $[a_j - \epsilon/2, b_j + \epsilon/2] \times \{Y_j\}$. Then every vertical interval in the image of M that has length $\geq \epsilon$ must contain a vertical interval v as above, hence must intersect the image of some B_j, hence must intersect $\mathbf{R}^2 \setminus \partial M$.

Note that missing the B_j's is an open condition. Hence, copies of M near this moved M can have no boundary interval of length $\geq \epsilon$.

The vertical decomposition of M, and the quotient continuum M'. Let V be a vertical line that intersects M. Let $\mathcal{G}(V)$ denote the set of components of $V \cap M$. Let $\mathcal{G} = \cup_V \mathcal{G}(V)$. Let $\mathcal{G}_0$ be the trivial extension of $\mathcal{G}$ to all of $\mathbf{R}^2$. (That is, $\mathcal{G}_0\setminus\mathcal{G}$ consists of the singleton sets of $\mathbf{R}^2\setminus M$.) Let $\pi : M \to (M' = M/\mathcal{G})$ and $\pi' : \mathbf{R}^2 \to (\mathbf{R}^2/\mathcal{G}_0)$ be the associated quotient maps.

Claim 1. The decomposition $\mathcal{G}_0$ is cellular and upper semicontinuous, so that $\mathbf{R}^2/\mathcal{G}_0$ is homeomorphic with $\mathbf{R}^2$ by the Moore Decomposition Theorem 2.3.1. Since each element of $\mathcal{G}$ intersects ∂M, $M' = \pi(M) = \pi'(M)$ is nowhere dense in $\mathbf{R}^2 \sim \mathbf{R}^2/\mathcal{G}_0$. Consequently M' is a 1-dimensional Peano continuum.

Proof (Claim 1). Since each element of $\mathcal{G}_0$ is a point or an arc $\mathcal{G} - 0$ is cellular. let $g_1, g_2, \ldots$ be elements of $\mathcal{G}_0$ containing convergent sequences $x_1, x_2, \ldots \to x$ and $y_1, y_2 \ldots \to y$, with $x_i, y_i \in g_i \in \mathcal{G}_0$. If $x \neq y$, then we must have g_i a vertical interval in M for all i sufficiently large. Thus x and y must be elements of M in the same vertical interval. The vertical intervals g_i join x_i to y_i. Hence their limits contain a vertical interval from x to y, which must lie in M. Thus x and y are in the same element of $\mathcal{G}_0$, and $\mathcal{G}_0$ is upper semicontinuous.

The remaining assertions of the claim are easily verified.

Claim 2. The projection map $\pi : M \to M'$ induces a map on fundamental groups that is injective. [This claim is the central assertion of Theorem 1.4.]

Proof (Claim 2). Let $f : \mathbf{S}^1 \to M$ be a continuous function such that $f' = (\pi \circ f) : \mathbf{S}^1 \to M'$ is nullhomotopic in M' (that is, there is a map $F' : \mathbf{B}^2 \to M'$ that extends f'). We must show that f is nullhomotopic in M. We may assume that f has been standardized in the sense that f restricted to $f^{-1}(\text{int}(M))$ is piecewise linear and nowhere vertical. Since ∂M contains no vertical interval by our previous adjustment, it follows that f is not vertical on any subinterval. These adjustments will allow us to give a rather precise analysis of the maps f and f'.

Analysis of $(f' = \pi \circ f) : \mathbf{S}^1 \to M'$.

Mapping Analysis Lemma. Suppose that $f' : \mathbf{S}^1 \to M'$ is a nullhomotopic mapping from the circle $\mathbf{S}^1$ into a 1-dimensional continuum M'. Then there is an upper semicontinuous decomposition H of $\mathbf{S}^1$ into compacta that has the following three properties:

(1) The mapping f' is constant on each element of H.

(2) The decomposition H is *noncrossing*. That is, if h_1 and h_2 are distinct elements of H, then the convex hulls $\mathbf{H}(h_1)$ and $\mathbf{H}(h_2)$ of h_1 and h_2 in the disk $\mathbf{B}^2$ are disjoint. [Equivalently, h_1 does not separate h_2 on $\mathbf{S}^1$.]

(3) The decomposition H is *filling*. That is, the disk $\mathbf{B}^2$ is the union of the convex hulls $\mathbf{H}(h)$ of the elements $h \in H$.

Proof. Let $F' : \mathbf{B}^2 \to M'$ be a map that extends $f' : \mathbf{S}^1 \to M'$. We define

$$H = \{\, h = C \cap \mathbf{S}^1 \mid \exists x \in M' \text{ such that } C \text{ is a component of } F^{-1}(x)\}.$$

It is obvious that H is an upper semicontinuous decomposition of $\mathbf{S}^1$ into compacta and that H satisfies property (1).

The proof of (2) is easy. If h_1 separates h_2 on $\mathbf{S}^1$, and if $h_1 = C_1 \cap \mathbf{S}^1$ and $h_2 = C_2 \cap \mathbf{S}^1$, then C_1 and C_2 must intersect, a contradiction.

The proof of (3) will require that we show that $\mathbf{S}^1/H$ is a contractible set. Assuming that $\mathbf{S}^1/H$ is contractible for the moment, we argue as follows. Let H' be the collection of sets in $\mathbf{R}^2$ that are either convex hulls $\mathbf{H}(h)$ of elements of $h \in H$ or are singleton sets that miss all such convex hulls. Since H is noncrossing, by (2), it follows easily that H' is a cellular, upper semicontinuous decomposition of $\mathbf{R}^2$. Let $\pi : \mathbf{R}^2 \to (\mathbf{R}^2/H' \approx \mathbf{R}^2)$ denote the projection map. If H were not filling, then the contractible set $\pi(\mathbf{S}^1) \approx \mathbf{S}^1/H$ would separate the nonempty sets $\pi(\mathbf{R}^2 \setminus \mathbf{B}^2)$ and $\pi(\mathbf{B}^2) \setminus \pi(\mathbf{S}^1)$ in $\mathbf{R}^2/H' \approx \mathbf{R}^2$, a contradiction.

The next paragraphs complete the proof by showing that $\mathbf{S}^1/H$ is contractible.

Each point $p \in \mathbf{B}^2$ lies on, or in a bounded complementary domain of, a unique such component C of maximal diameter. We may redefine F' so that $F'(p) = F'(C)$. This modification does not alter the decomposition H. After this modification, the nondegenerate components form the nondegenerate elements of a cellular decomposition G of $\mathbf{R}^2$; and, by the Moore Decomposition Theorem 2.3.1, the quotient $\mathbf{R}^2/G$ is homeomorphic with $\mathbf{R}^2$. We denote the quotient map by $\pi : \mathbf{R}^2 \to (\mathbf{R}^2/G \approx \mathbf{R}^2)$. The modified F' factors through the projection $\pi|\mathbf{B}^2 : \mathbf{B}^2 \to \mathbf{B}^2/(G|\mathbf{B}^2)$:

$$F' : \mathbf{B}^2 \xrightarrow{\pi|\mathbf{B}^2} \mathbf{B}^2/(G|\mathbf{B}^2) \xrightarrow{F''} X.$$

The image $\pi(\mathbf{B}^2)$ of the disk $\mathbf{B}^2$ is contractible because it is a strong deformation retract of the disk $\pi(2\mathbf{B}^2) \subset \mathbf{R}^2/G$. [The set $\pi(2\mathbf{B}^2)$ is a disk since it is a compact set in the plane $\mathbf{R}^2/G$ whose boundary is a simple closed curve.]

The image $\pi(\mathbf{B}^2)$ of the disk $\mathbf{B}^2$ is 1-dimensional since (i) it admits the mapping $F'' : \pi(\mathbf{B}^2) \to M'$ into a 1-dimensional space M and the point preimages of F'' are totally disconnected, while (ii) a map that reduces dimension by dimension k must have at least one point preimage of dimension k [7, Theorem VI 7].

The images $\pi(\mathbf{B}^2)$ and $\pi(\mathbf{S}^1)$ are equal for the following reasons. Since $\pi(\mathbf{B}^2)$ is compact and 1-dimensional, the open set $\pi(\mathbf{R}^2 \setminus \mathbf{B}^2)$ is dense in the plane $\mathbf{R}^2/G$. Hence the image of $\pi(\mathbf{R}^2 \setminus \mathrm{int}(\mathbf{B}^2))$ is the entire plane. Consequently, $\pi(\mathbf{S}^1) \supset \pi(\mathbf{B}^2)$. The opposite inclusion is obvious. We conclude that $\pi(\mathbf{S}^1)$ is contractible.

This completes the proof of the Mapping Analysis Lemma.

Completion of the proof that $f : \mathbf{S}^1 \to M$ is contractible.

We recall the cellular, upper semicontinuous decomposition $\mathcal{G}$ of $\mathbf{R}^2$ that has as its nondegenerate elements the maximal vertical intervals in M and whose quotient map $pi : \mathbf{R}^2 \to \mathbf{R}^2/\mathcal{G}$ takes M onto M'. We use the Mapping Analysis Lemma to obtain an upper semicontinuous decomposition H of $\mathbf{S}^1$ that models the shrinking of $f' = (\pi \circ f) : \mathbf{S}^1 \to M'$ in the 1-dimensional set M'. Since the decomposition H is noncrossing and filling, we may expand this decomposition H to a decomposition G of $\mathbf{B}^2$ by taking as elements the convex hulls in $\mathbf{B}^2$ of the elements of H. The shrinking of f in M will rely on the interplay between the decompositions $\mathcal{G}$ and G. We shall use the decomposition G of $\mathbf{B}^2$ as a model on which we shall base the construction of a continuous function $F : \mathbf{B}^2 \to M$ that extends $f : \mathbf{S}^1 \to M$.

If, for each $g \in G$, $f|g \cap \mathbf{S}^1$ were constant (as is true for f'), we could simply define $F(g) = f(g \cap \mathbf{S}^1)$. However, this need not be the case. All that we know is that $\forall g \in G$, $\exists h(g) \in \mathcal{G}$ such that $f(g \cap \mathbf{S}^1) \subset h(g)$. We need to show how to define $F|g : g \to h(g) \subset M$ in such a way that the union $F = \cup\{F|g : g \in G\}$ is a continuous extension of f.

If g is a single point, then that point lies in $\mathbf{S}^1$, and we may define $F(g) = f(g)$.

If g is an interval with its ends in $\mathbf{S}^1$, then we extend the map $f|\partial g$ linearly to all of g.

If g is a disk, then we use an *ideal triangulation* of g in the following way:

The set g is the convex hull $\mathbf{H}(h)$ of a closed subset h of the unit circle $\mathbf{S}^1$. Since g is a disk, h contains at least three points. An *ideal triangle* is a triangle in $\mathbf{B}^2$ that has its vertices on $\mathbf{S}^1$. A collection $\{T_i\}$ of ideal triangles is said to be an ideal triangulation of the convex hull g provided that the triangles have disjoint interiors, have vertices in h, and have union whose intersection with $\mathrm{int}(\mathbf{B}^2)$ is precisely $g \cap \mathrm{int}(\mathbf{B}^2)$.

Triangulation Lemma. If $g = \mathbf{H}(h)$ is a disk, then g has an ideal triangulation.

Proof (Triangulation Lemma). Every point $x \in \mathbf{H}(h) \cap \mathrm{int}(\mathbf{B}^2)$ has a neighborhood in $\mathbf{H}(h)$ that is in the convex hull of a finite collection of points in h. [Hint: every point of a convex hull lies in the hull of a finite subset; consider separately the case where the point is in the interior or on the boundary of such a finite polygon.] Hence, every compact subset of $\mathbf{H}(h) \cap \mathrm{int}(\mathbf{B}^2)$ is in the convex hull of a finite collection of points in h.

Let $C_1 \subset C_2 \subset C_3 \subset \cdots$ be an exhaustion of $\mathbf{H}(h) \cap \mathrm{int}(\mathbf{B}^2)$ by compact sets, and let $F_1 \subset F_2 \subset F_3 \subset \cdots$ be finite subsets of h such that $C_i \subset \mathbf{H}(F_i)$. It suffices to show that any ideal triangulation T_i of $\mathbf{H}(F_i)$ extends to an ideal triangulation T_{i+1} of $\mathbf{H}(F_{i+1}$, for then we may take $T(X) = \cup_{i=1}^{\infty} T_i$.

To extend T_i to T_{i+1}, it suffices to see that we can add one point p at a time to F_i. Since each edge of T_i separates $\mathbf{B}^2$, the domain of $\mathbf{B}^2 \setminus |T_i|$ that contains p is bounded by a single edge rs of T_i followed by an arc of $\mathbf{S}1$ that contains p. We simply add the triangle prs to T_i.

This completes the proof of the Triangulation Lemma. With the Triangulation Lemma in hand, we are ready to define $F|g : g \to M$, for the case where g is a disk.

In this case, we note that $h = g \cap \mathbf{S}^1$ is a compact, totally disconnected set having at least three points. Hence, by the Triangulation Lemma, g has an ideal triangulation $T(g)$. We define F on $g \cap \mathbf{S}^1$ to equal f. On each triangle t_i of $T(g)$, we define F to be the linear extension of f restricted to the three vertices of t_i.

Proof that $F|g$ is continuous for each $g \in G$. If $F|g$ is not continuous, then $\exists x_1, x_2, \cdots \to x$ in g and $\epsilon > 0$ such that, $\forall i$, $d(F(x_i), F(x)) \geq \epsilon$. Since $F|g \cap \mathbf{S}^1 = f|g \cap \mathbf{S}^1$ is continuous, we may assume that each x_i lies in $\mathrm{int}(\mathbf{B}^2)$. Since F is continuous on any finite union of triangles of $T(g)$ and since $T(g)$ is locally finite in $\mathrm{int}(\mathbf{B}^2)$, we may assume that $x \in h = g \cap \mathbf{S}^1$ and that $x_1, x_2, \ldots$ come from distinct triangles of $T(g)$. Since these triangles cannot accumulate at any interior point of $\mathbf{B}^2$, they must, in fact, have diameter going to 0 and approach x. But then their vertices approach x and, by linearity, their images approach x, a contradiction. Hence $F|g$ is continuous.

Proof that F is continuous. If F is not continuous, then $\exists\, x_1, x_2, \cdots \to x$ in $\mathbf{B}^2$ and $\epsilon > 0$ such that, $\forall\, i$, $d(F(x_i), F(x)) \geq \epsilon$. Since, $\forall\, g \in G$, $F|g$ is continuous, we may assume that $x_1, x_2, \ldots, x$ all come from distinct elements $g_1, g_2, \ldots, g$ of G. By continuity of $F|\mathbf{S}^1 = f$, we may ignore those x_i in $\mathbf{S}^1$. Hence, we may assume that $x_i \in \mathrm{int}(\mathbf{B}^2)$, that g_i is either an arc t_i or a disk, one of whose triangles t_i contains x_i. If the t_i approach x, then $x \in \mathbf{S}^1$, the vertices of the t_i approach x, and the images of the t_i approach $F(x)$ by linearity and the continuity of $F|\mathbf{S}^1 = f$. Otherwise, we may assume that the t_i approach an edge t of g that contains x. Again, their vertices approach the vertices of t, and the continuity of $F|\mathbf{S}^1 = f$ and linearity imply that $F(x_i) \to F(x)$, a contradiction. We conclude that F is continuous.

This completes the proof of Theorem 1.4. We recall the corollary and question associated with Theorem 1.4:

Corollary 1.4.1. If M is a planar Peano continuum, then the fundamental group of M embeds in an inverse limit of finitely generated free groups.

Proof. This theorem is well-known for 1-dimensional continua. See, for example, [6] and [2].

Question 1.4.2. If M is a planar Peano continuum whose fundamental group is isomorphic with the fundamental group of some 1-dimensional planar Peano continuum, is it true that M is homotopically 1-dimensional?

It is not difficult to see that the projection that we have given that takes M onto M' does not give a surjection on fundamental group if M is not homotopically 1-dimensional. The key issue to resolve here is whether an arbitrary group embedding into the group of a 1-dimensional continuum can always be induced by a continuous map.

We add two final corollaries.

Corollary 1.4.3. If M is a planar Peano continuum, $f : \mathbf{S}^1 \to M$ is a loop in M, and f is nullhomotopic in every neighborhood of M in $\mathbf{R}^2$, then f is nullhomotopic in M.

Proof. It follows easily that $f' : \mathbf{S}^1 \to M'$ is nullhomotopic in each neighborhood of M' in $\mathbf{R}^2$. But it is well-known [6],[2] that this implies that f' is nullhomotopic in M'. Thus the argument of Theorem 1.4 applies to show that f is nullhomotopic in M.

Corollary 1.4.4. If M is a planar Peano continuum, then $\pi_1(M)$ embeds in an inverse limit of free groups. Those free groups may be taken to be the fundamental groups of standard neighborhoods (disks with holes) of M in $\mathbf{R}^2$.

Proof. Corollary 1.4.4 is a standard corollary to Corollary 1.4.3.

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